Lecture 2

extremal graph theory : "general property" to controle "local structure"

$e(G)$ — spanning structure — Δ, K_{r+1}

$\delta(G)$. — C_{2k+1}

$ex(n, H) = \max\{e(G) : |G| = n,\ H \not\subseteq G\}$.

Turán's theorem : $ex(n, K_{r+1}) \leq (1 - \frac{1}{r})\frac{n^2}{2}$

structure :

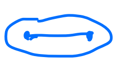
① perfect matching

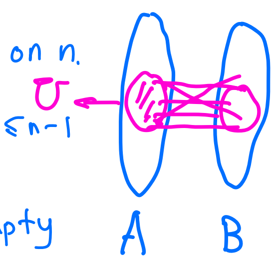
matching 

A perfect matching in a bipartite graph with two sets of equal size is a collection of edges such that every vertex is contained in exactly one of them.

Th1. (Hall's theorem) Let G be a bipartite graph with parts A and B of equal size. If $\forall\, U \subset A$, $|N_G(U)| \geq |U|$, then G contains a pM.

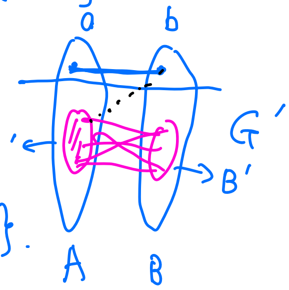
proof. Let $|A| = |B| = n$. We prove it by induction on n. $|A| = |B|$

$n = 1$ ✓.  We assume that it is true for $\leq n-1$ and prove it on n.



Case 1. If $|N_G(U)| \geq |U| + 1$, for every non-empty proper subset U of A, then we choose an edge $\{a, b\}$ of G.

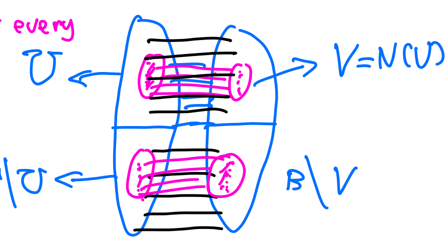
Let $G' = G - \{a, b\}$. Then every non-empty set $U \subset A \setminus \{a\}$ satisfies $|N_{G'}(U)| \geq |N_G(U)| - 1 \geq |U|$.



Therefore, there is a pM between $A \setminus \{a\}$ and $B \setminus \{b\}$.
Adding $\{a, b\}$ give a pM in G.

Case 2. $\exists$ non-empty proper subset U of A such that $|N_G(U)| = |U|$.

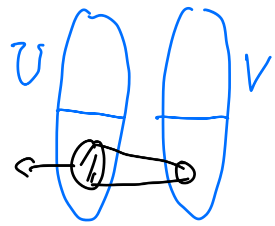
Let $V = N(U)$. By induction. (Hall's condition holds for every subset of U),
there is a perfect matching between U and V.



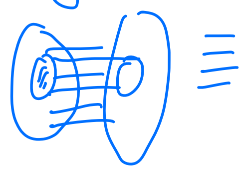
Hall's condition holds between $A \setminus U$ and $B \setminus V$??

If not, there would be some W in $A \setminus U$ with fewer than $|W|$

If not, there would be some W in $A \setminus U$ with fewer than $|W|$ neighbors in $B \setminus V$. Then $W \cup U$ would be a subset of A with fewer than $|W \cup U|$ neighbors in B, which is a contradiction. Therefore, there is a pm between $A \setminus U$ and $B \setminus V$.



Then we get a pm in the whole graph by putting them together.



hypergraph:

vertex: $\{v_1, v_2, \cdots, v_n\}$

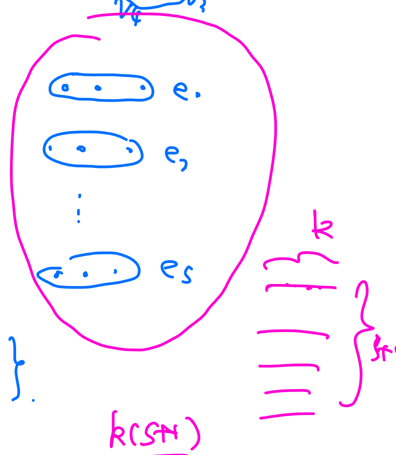
edge $v_1 v_2$, $v_1 v_3$

k-uniform hypergraph   3-uniform $v_1 v_2 v_3$

Erdős Matching Conjecture

H: k-uniform hypergraph

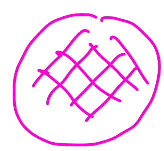
$\nu(H)$: "matching number": the maximum size of a matching in H.

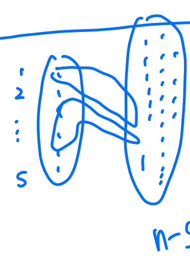


$m(n, k, s)$: $\max\{e(H) : |V(H)| = n,\ \nu(H) < s+1\}$ (k-uniform)

Erdős Matching Conjecture

$m(n, k, s) = \max\left\{\binom{n}{k} - \binom{n-s}{k},\ \binom{k(s+1)-1}{k}\right\}$

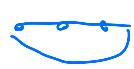




Extremal set theory.

$\boxed{k=3\ \checkmark}$

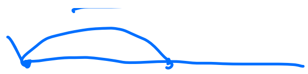
$S=1$. Erdős-Ko-Rado Theorem "intersecting family"



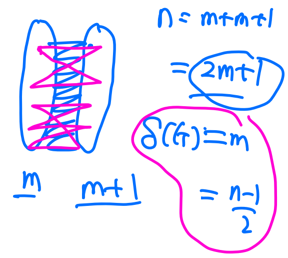
Theorem Hamilton cycle

A Hamiltonian cycle in a graph G is a cycle which visits every vertex exactly once.

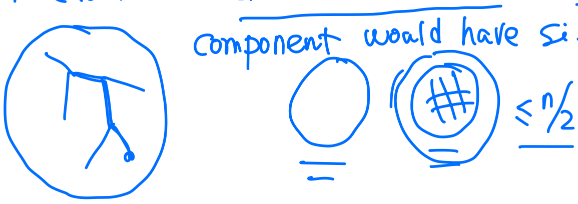
① $\delta(G) \geq 2$. $\Rightarrow$ G contains a cycle.

$\delta(G) \geq 2$  longest path.

Thm. (Dirac's theorem) If a graph G satisfies $\delta(G) \geq n/2$, where $n = |V(G)|$, then it contains a Hamiltonian cycle.

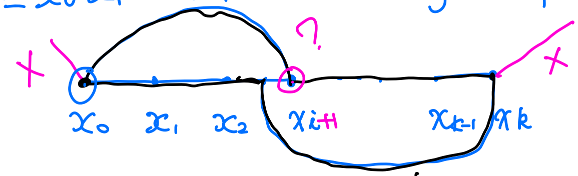


proof. Claim 1. G is connected. if not, the smallest component would have size at most $n/2$ and



no vertex in this component could have degree $n/2$ or more.

Let $P = x_0 x_1 \cdots x_k$ be a longest path in G.

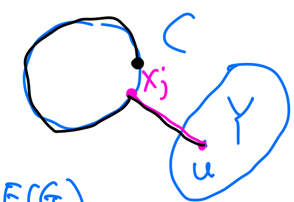


Claim 2. $\exists$ a cycle visiting $x_0, x_1, \cdots, x_k$.

there must be some i for which both $x_0 x_{i+1}$, and $x_i x_k$ are edges in G

Claim 3. $C = x_0 x_{i+1} x_{i+2} \cdots x_k x_i x_{i-1} \cdots x_0$ is a Hamiltonian cycle.

Suppose not, there is a set of vertices Y which are not contained in C. ~~Since G is connected~~,



$|C| \geq \frac{n}{2} + 1$. $\forall u \in Y$, $d(u) \geq n/2$

$\exists$ a vertex $x_j \in V(C)$ such that $ux_j \in E(G)$,

We may define a path P' starting at y, going to x_j and then around the cycle C which is longer than P, This would contradict with our assumption about P. $\square$

tree: "connected" + "no cycle"

Erdős–Sós conjecture $\qquad \frac{d(G) \times n}{2} = e(G)$

If a tree T has t edges, then any graph G ~~with average~~

degree t must contain a copy of Γ.

[Ajtai – Komlós – Simonovits – Szemerédi] proved it for sufficiently large graph G.

Theorem 3. If a graph G has average degree $2t$, then it contains every tree T with t edges.

proof. claim 1 Every graph of average degree $2t$ has a subgraph of minimum degree t.

$G \quad d(G) = 2t \qquad e(G) \geqslant nt = \frac{d(G)\,n}{2} = \frac{2t\cdot n}{2} = nt.$

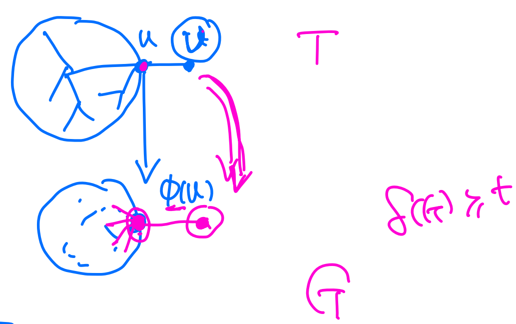
$\delta(G) \geqslant t$ ✓.

$\exists\, v \quad d(v) < t.$ delete v $\qquad G' = G - v \qquad d(G') \geqslant \frac{[nt - d(v)]\times 2}{n-1}$

$= \frac{2nt - 2d(v)}{n-1} \overset{?}{\geqslant} 2t \Leftarrow 2nt - 2d(v) \geqslant 2tn - 2t$ ✓

If there is a vertex of degree less than t, delete it.
This will not decrease the average degree. Moreover, the process must end, since any graph with fewer than $2t$ vertices cannot have average degree $2t$.

Claim 2. We can embed any tree with t edges into a graph with minimum degree t.



Homework: Remove TV control needs two working batteries.
There is a bag of 15 batteries with 5 working batteries and 10 dead ones. How many times we have to try to guarantee to find a working pair of batteries?

Th (Schur's theorem) $\qquad x+y=z \qquad x+y=2z$

$\forall$ every positive integer r, $\exists\ N=N(r)$, such that if each element of $[N]$ is colored using one of r colors, then there is a monochromatic solution to $x+y=z$.

$[N]=\{1, 2, \dots N\}$

$\downarrow \ \downarrow \ \downarrow$

$R \ B \ Y$

$x+y=z$: $R \ R \ R$

Ramsey's theorem

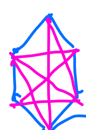
"order from chaos"

Complete disorder is impossible.

Theorem (Multicolor triangle Ramsey Theorem)

$\forall\ r$, $\exists\ N=N(r)$, such that if each edge of K_N is colored using one of r colors, then there is a monochromatic triangle.

$R(3,3)=6$



proof. let $N_1=3$, and $N_r = r(N_{r-1}-1)+2$ for all $r \geq 2$.

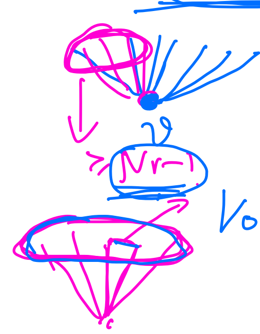
We prove it by induction on r. In fact we will prove every coloring of the edges of K_{N_r} by r colors has a monochromatic triangle.

$r=1 \quad N_1=3$ ✓ Suppose the claim is true for $r-1$ colors.

Consider any edge-coloring of K_{N_r} using r colors.

Choose a vertex v $\quad d(v) = N_r - 1 = r(N_{r-1}-1)+2-1 = r(N_{r-1}-1)+1$

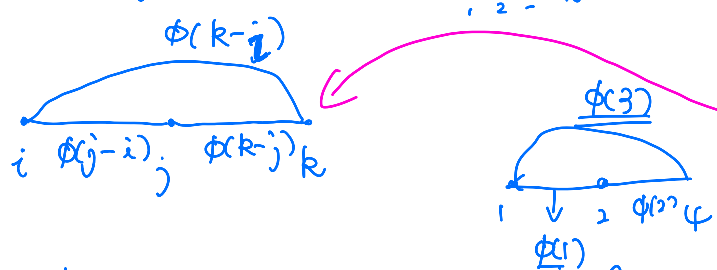
$\dfrac{r(N_{r-1}-1)+1}{r} \geq N_{r-1}$, of at least N_{r-1} edges incident to v has the same color, Say Red.



If there is a red edge inside V_0, then we obtain a red triangle. Otherwise there are at most $(r-1)$ colors appearing among $|V_0| \geq N_{r-1}$ vertices. Then we have a monochromatic triangle inside V_0 by the induction hypothesis

proof of Schur's theorem. Let $\phi : [N] \to [r]$ be a coloring.
Color the edges of a complete graph with vertices $\{1, 2, \cdots, N+1\}$ by giving the edges $\{i, j\}$ with $i<j$ the color $\phi(j-i)$.

K_{N+1}



By multicolor triangle Ramsey theorem, if N is large enough, then there is a monochromatic triangle, say on vertices $i<j<k$.
So $\underset{z}{\phi(k-i)} = \underset{x}{\phi(j-i)} = \underset{y}{\phi(k-j)}$. Take $x = j-i$, $y = k-j$, $z = k-i$. Then $\phi(x) = \phi(y) = \phi(z)$ and $x+y=z$ $(j-i+k-j = k-i)$ as desired.

Open problem. Is there a constant $C>0$, such that if $N \geq C^r$, then every edge coloring of K_N using r colors contains a monochromatic triangle? (upper bound)

$\boxed{\text{Proposition}}$ (lower bound) (Homework 2)
For every positive integer r, there exists an edge coloring of K_{2^r} using r colors with no monochromatic triangles.
$N = 2^r$ $\quad < C^r$?

(Erdős-Stone-Simonovits theorem) $\forall H$

$$ex(n, H) \leq \left(1 - \frac{1}{\chi(H)-1}\right) \times \frac{n^2}{2} + o(n^2)$$

H
C_{2k}
C_{2k+1}